B.Sc. (Part—III) Semester—V Examination
MATHEMATICS (NEW)
Paper—IX
(Mathematical Analysis)

Time : Three Hours] [Maximum Marks : 60

Note :— (1) Question No. **1** is compulsory and attempt it once only.

(2) Attempt **ONE** question from each unit.

1. Choose the correct alternative :

(i) Let P = (1, 3, 4, 5, 6) be a partition of [1, 6] and if $M_1 = 2$, $M_2 = 3$, $M_3 = 4$, $M_4 = 5$ are lub's of F then U(P, F) is : 1

(a) 11 (b) 10

(c) 12 (d) 16

(ii) Let f be a bounded function defined on [a, b] and P be any partition of [a, b], P* be refinement of P. Then L(P, f) and L(P*, f) satisfy : 1

(a) $L(P, f) \le L(P^*, f)$ (b) $L(P, f) \ge U(P^*, f)$

(c) $L(P, f) \ge L(P^*, f)$ (d) None of these

(iii) An improper integral $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ converges to : 1

(a) $\sqrt{x}$ (b) $-x$

(c) x (d) 0

(iv) An integral $\int_0^{\infty} e^{-kx} x^{n-1}\,dx$ is : 1

(a) $k^n \overline{|n}$ (b) $\frac{\overline{|n}}{k^n}$

(c) k^n (d) $\overline{|n}$

(v) If a function f(z) = u(x, y) + iv(x, y) is Analytic in a region D, then : 1

(a) $u_x = u_y$ and $u_y = v_x$ (b) $u_x = -u_y$ and $u_y = -v_x$

(c) $u_x = v_y$ and $u_y = -v_x$ (d) None of these

(vi) If $w = u + iv$ is analytic function in D, then $\frac{dw}{dz}$ is : 1

(a) $-\frac{\partial w}{\partial x}$ (b) $\frac{\partial w}{\partial y}$

(c) $-\frac{\partial w}{\partial y}$ (d) $\frac{\partial w}{\partial x}$

(vii) A Mobius transformation $w = az$, a is real number, is : 1

(a) Rotation transformation (b) Magnification transformation

(c) Translation transformation (d) None of these

(viii) A bilinear transformation with only one fixed point is : 1

(a) Loxodromic (b) Parabolic

(c) Elliptic (d) Hyperbolic

(ix) For any collection of $\{A_\alpha\}$ open sets, $\bigcup_\alpha A_\alpha$ is : 1

(a) Closed (b) Open

(c) Semi-open (d) None of these

(x) A metric space (X, d) is complete if every Cauchy sequence in X is : 1

(a) Bounded (b) Unbounded

(c) Convergent (d) Divergent

UNIT—I

2. (a) Prove that a bounded function f defined on [a, b] is integrable on [a, b] iff for any $\epsilon > 0$ there exist a $\delta > 0$ such that for every partition P of [a, b] with $\mu(P) < \delta$, $U(P, f) - L(P, f) < \epsilon$. 5

(b) If f is function defined by $f(x) = x$ on [0, 2], then show that f is integrable in Riemann sense over [0, 2] and $\int_0^2 f(x)\,dx = 2$. 5

3. (a) Prove that if f is continuous and integrable on [a, b], then $\int_a^b f(x)\,dx = f(c)(b-a)$ where c is some point in [a, b]. 5

(b) Let the function f be defined as $f(x) = \begin{cases} 1 & , x \text{ is rational} \\ -1, & x \text{ is irrational} \end{cases}$. Show that f is not R-integrable on [0, 1]. But $|f| \in R[0, 1]$. 5

UNIT—II

4. (a) Prove that $\int_{a+}^{b} \frac{1}{(x-a)^p} dx$ converges if $p < 1$ and diverges if $p \geq 1$. 4

(b) Test the convergence of $\int_{2}^{\infty} \frac{1}{\sqrt{x^2 - 1}} dx$. 3

(c) Show that $\int_{1}^{\infty} \frac{e^{-x}}{x} dx$ is convergent. 3

5. (a) Prove that :

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx.$$ 4

(b) Evaluate :

$$\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx.$$ 3

(c) Prove that :

$$\int_0^\infty e^{-kx} x^{n-1} dx = \overline{|n} / k^n.$$ 3

UNIT—III

6. (a) Prove that if $f(z) = u(x, y) + iv(x, y)$ is analytic function in region D, then $u_x = v_y$ and $u_y = -v_x$ in D. 5

(b) Prove that $u = y^3 - 3x^2y$ is a harmonic function. Find its conjugate and the corresponding analytic function $f(z)$ in terms of z. 5

7. (a) If the function $f(z) = u + iv$ is analytic in the domain D, then prove that family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ form an orthogonal system, where c_1 and c_2 are constants. 5

(b) Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at the origin, although C-R equations are satisfied at origin. 5

UNIT—IV

8. (a) Prove that cross-ratio remains invariant under a bilinear transformation. 5

(b) Let D be a region in z-plane is bounded by $x = 0$, $y = 0$, $x = 2$ and $y = 1$. Find the region in w-plane into which D is mapped under the transformation $w = z + (1 - 2i)$. 5

9. (a) Prove that every bilinear transformation with two non-infinite fixed points p, q can be put in the normal form $\frac{w-p}{w-q} = k\left(\frac{z-p}{z-q}\right)$ where k is constant. 5

(b) Find the fixed points of the transformation $w = \frac{z-1}{z+1}$; state whether it is elliptic, hyperbolic or Loxodromic. Find also its normal form. 5

UNIT—V

10. (a) If p is a limit point of a set A, then prove that every neighbourhood of p contains infinitely many points of A. 4

(b) Show that $d(x, y) = |x - y|, \forall x, y \in R$ defines a metric on R. 4

(c) Define :

(i) Open set

(ii) Closed set. 2

11. (a) Define a Cauchy sequence in a metric space and prove that every convergent sequence in a metric space is a Cauchy sequence. 4

(b) Prove that for any finite collection $A_1, \ldots, A_n$ of open sets $\bigcap_{i=1}^{n} A_i$ is open set. 4

(c) Define a metric d on a space X. 2