B.A./B.Sc. (Part–III) Semester–VI Examination

MATHEMATICS–XI

(Linear Algebra)

Time—Three Hours] [Maximum Marks—60

Note :—(1) Question No. 1 is compulsory. Solve this question in **one** attempt only.

(2) Attempt **ONE** question from each Unit.

1. Choose the correct alternatives :

(i) If U and W are the subspaces of a vector space V(F) then $U \cup W$ is a subspace iff :

(a) $U \subseteq W$ or $W \subseteq U$

(b) $U \supseteq W$ and $W \supseteq U$

(c) $U \cap W = \{0\}$

(d) None of these 1

(ii) Any super set of a linearly dependent set is :

(a) Linearly independent

(b) Linearly dependent

(c) Linearly independent or linearly dependent

(d) None of these 1

(iii) If W is a subspace of a finite dimensional vector space V then dim (V/W) =

(a) dim (V/W) (b) dim V – dim W

(c) dim V + dim W (d) None of these 1

(iv) If $T : U \to V$ be a linear map then R(T) is a subspace of :

(a) U (b) $U \cap V$

(c) V (d) None of these 1

(v) Let $T : V_2 \to V_2$ be a linear map and dim R(T) = 2. Then N(T) is :

(a) V_1 (b) V_2

(c) $\{0_{V_2}\}$ (d) None of these 1

(vi) Annihilator of W, A(W) is a subspace of :

(a) W (b) V

(c) $\hat{V}$ (d) None of these 1

(vii) Every set of orthogonal vectors is :

(a) Linearly independent

(b) Linearly dependent

(c) Linearly independent and linearly dependent

(d) None of these 1

UNIT–V

10. (a) Let M be an R module. Then prove the following :

(i) $r \cdot 0 = 0, \ \forall\, r \in R$

(ii) $-(r \cdot a) = r(-a) = (-r)a, \ \forall\, r \in R$ and $m \in M$. 4

(b) Prove that arbitrary intersection of submodules of a module is a submodule. 3

(c) Let A be a submodule of unital R module M, prove that M/A is also unital R-module. 3

OR

11. (p) Let T be a homomorphism of an R-module M to an R-module H. Prove that :

(i) K(T) is a submodule of M and R(T) is a submodule of H; and

(ii) T is $1-1 \Leftrightarrow K(T) = \{0\}$. 5

(q) If A is a submodule of an R-module M and $T : M \to M/A$ defined by $T(m) = A + m, \ \forall\, m \in M$, then prove that T is an R-homomorphism of M into M/A and Ker T = A. 5

(q) Let U and V be finite dimensional complex vector spaces and $A : U \to V$; $B : U \to V$ be linear maps, then prove that $(A + B)^* = A^* + B^*$. 5

UNIT–IV

8. (a) Prove that in an inner product space V,

   (i) $\|\alpha \cdot u\| = |\alpha| \cdot \|u\|$

   (ii) $\|u + v\| \leq \|u\| + \|v\|$, $\alpha \in F$ and $u, v \in V$. 4

   (b) If V is a finite dimensional inner product space and W is a subspace of V then $(W^{\perp})^{\perp} = W$. Prove this. 3

   (c) Using Gram-Schmidt orthogonalisation process, orthonotmise the linearly independent subset $\{(1, 1, 1), (0, 1, 1) (0, 0, 1)\}$ of V_3. 3

OR

9. (p) Prove that every finite dimensional inner product space has an orthogonal basis. 4

   (q) Let V be an inner product space V and $u, v \in V$, prove that :

   $$|u \cdot v| \leq \|u\| \cdot \|v\|$$ 3

   (r) In an inner product space V over F, prove the parallelogram law :

   $$\|u + v\|^2 + \|u - v\|^2 = 2 (\|u\|^2 + \|v\|^2)$$ 3

(viii) If V is a finite dimensional vector space and W is a subspace of V then $\left(W^{\perp}\right)^{\perp} =$

   (a) V (b) W

   (c) $W^{\perp}$ (d) None of these 1

(ix) R-Module homomorphism is a linear transformation if :

   (a) R with unit element

   (b) R is commutative

   (c) R is a field

   (d) None of these 1

(x) If the ring R has a unit element 1 and $1 \cdot a = a$, for all $a \in M$, then M is called :

   (a) A unital R module

   (b) Right R-module

   (c) Left R-module

   (d) None of these. 1

UNIT–I

2. (a) Let R^+ be the set of all positive real numbers. Define the operations of addition $\oplus$ and scalar multiplication $\otimes$ as follows :

   $$u \oplus v = uv, \quad \forall\, u, v \in R^+$$

   and $\alpha \otimes u = u^{\alpha}$, $\forall\, u \in R^+$ and $\alpha \in R$

   Prove that R^+ is a real vector space. 3

(b) Let U and W be two subspaces of a vector space V and Z = U + W. Then prove that $Z = U \oplus W \Leftrightarrow z = u + w$ is unique representation for any $z \in Z$ and for some $u \in U$, $w \in W$. 4

(c) If $\{v_1, v_2, \ldots v_n\}$ is a basis for V or span V over F and if $w_1, w_2, \ldots w_m \in V$ are linearly independent over F then prove that $m \leq n$. 3

OR

3. (p) Prove that a nonempty subset U of a vector space V over F is a subspace of V iff :

(i) $u + v \in U, \forall u, v \in U$ and

(ii) $\alpha u \in U, \forall \alpha \in F, u \in U$. 4

(q) Prove that an arbitrary intersection of subspaces of a vector space is again a subspace. 3

(r) Extend the linearly independent set {(1, 1, 1, 1), (1, 2, 1, 2)} in V_4 to a basis for V_4. 3

UNIT–II

4. (a) Let $T : U \to V$ be a linear map then prove that :

(i) R(T) is a subspace of V

(ii) T is one-one $\Leftrightarrow N(T) = \{0_U\}$, a subspace of U. 4

(b) Determine range, rank, kernel and nullity of the linear map $T : V_3 \to V_4$ defined by :

$$T(x_1, x_2, x_3) = (x_1, x_1 + x_2, x_1 + x_2 + x_3, x_3).$$ 4

(c) Prove that, a square matrix is non-singular iff its column Vectors are linearly independent. 2

OR

5. (p) Let $T : U \to V$ be a linear map and U be a finite dimensional vector space, then :

$$\dim R(T) + \dim N(T) = \dim V.$$ Prove this. 4

(q) Let $T : U \to V$ be a linear map which is nonsingular then prove that $T^{-1} : V \to U$ is linear, one-one and onto. 3

(r) Let $T : V_3 \to V_3$ be a linear map defined as :

$$T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$$

Then show that $(T^2 - I).(T - 3I) = 0$. 3

UNIT–III

6. (a) Let V be a finite dimensional vector space over F. Then prove that $V = \hat{\hat{V}}$. 5

(b) Define annihilator of W. Prove that annihilator of $W = A(W)$ is a subspace of $\hat{V}$. 5

OR

7. (p) If S is a subset of vector space V and

$$A(s) = \left\{ f \in \hat{V} / f(s) = 0 \ \forall s \in S \right\}$$

Then prove that $A(s) = A(L(s))$, where L(s) is the linear span of S. 5