10. (c) Let V be a non-zero finitely generated vector space over a field F. Then prove that V admits a finite basis. 8

(d) Prove that the submodules of the quotient module $\frac{M}{N}$ are of the form $\frac{U}{N}$, where U is a submodule of M containing N. 8

AQ - 806

First Semester M. Sc. (Part - I) (C.B.C.S.) Examination

(New)

MATHEMATICS

Paper - 2 MTH - 102

(Advanced Abstract Algebra - I)

P. Pages : 6

Time : Three Hours] [Max. Marks : 80

Note : Solve any **One** question from each unit.

UNIT - I

1. (a) Define : Nilpotent group.
Prove that a group G is nilpotent iff G has a normal series. 8

(b) Define :

(i) Normal subgroup.

(ii) Quotient group.

Show that a subgroup H of G is normal iff $Ha \neq Hb \Rightarrow aH \neq bH$. 8

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2. (c) If G is the additive group of reals and N is the subgroup of G consisting of integers, prove that G/N is isomorphic to the group H of all complex numbers of absolute value 1 under multiplication. 8

(d) Define : Solvable group.

Prove that a group G is solvable iff G has a normal series with abelian factors. Further, a finite group is solvable iff its composition factors are cyclic groups of prime orders. Prove this. 8

UNIT – II

3. (a) Let A be a finite abelian group.

Prove that there exists a unique list of integers $m_1, m_2, \ldots, m_k$ (all > 1)

Such that

$|A| = m_1 \ldots m_k$, $m_1 | m_2 | \ldots | m_k$,

and $A = C_1 \oplus \ldots \oplus C_k$,

where $C_1, \ldots, C_k$ are cyclic subgroups of A of order $m_1, \ldots, m_k$ respectively.

Consequently,

$$A \simeq Z_{m_1} \oplus \ldots \oplus Z_{mk}$$ 8

(b) Define : p – group.

Let A be a finite abelian group and let p be a prime. If p divides |A|, then show that A has an element of order p. 8

4. (c) Let G be a finite group, and let p be a prime. Then prove that all sylow p–subgroups of G are conjugate, and their number η_p divides O(G) and satisfies $\eta_p \equiv 1 \pmod{p}$. 8

(d) Find all the non–isomorphic abelian groups of order.

(i) 8 (ii) 6

(iii) 20 (iv) 25 8

UNIT – III

5. (a) Prove that for any ring R any ideal $A \neq R$, the following are equivalent,

(i) A is maximal.

(ii) The quotient ring R/A has no non–trivial ideals.

(iii) For any element $x \in R$, $x \notin A$, $A + (x) = R$. 8

(b) Prove that

(i) $\frac{A+B}{A} \cong \frac{B}{A \cap B}$,

where A, B are ideals of Ring R.

(ii) $\frac{\mathbb{Z}}{\langle 2 \rangle} \cong \frac{5\mathbb{Z}}{10\mathbb{Z}}$ 8

6. (c) Let R be a commutative ring. Prove that an ideal P of R is prime ideal iff for two ideals A, B of R, $AB \subseteq P$ implied either $A \subseteq P$ or $B \subseteq P$. 8

(d) Let R be a commutative ring with unity and let I_1 and I_2 be two ideals of R then show that

(i) $\phi : R \longrightarrow \frac{R}{I_1} x \frac{R}{I_2}$, s. t. $\phi(x) = (x + I_1, x + I_2)$ is homomorphism s. t. Ker $\phi = I_1 \cap I_2$.

(ii) I_1 and I_2 are comaximal ideals iff ϕ is onto. 8

UNIT – IV

7. (a) Prove that every Principal Ideal Domain (PID) is a Unique Factorization Domain (UFD) but a UFD is not necessarily a PID. 8

(b) Define :

(i) Associates.

(ii) Irreducibles.

(iii) Primes.

Prove that in a PID, an element is an irreducible iff it is prime. 8

8. (c) Define : Euclidean Domain.
Prove that the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \,/\, a, b \in \mathbb{Z}\}$ is Euclidean domain. 8

(d) Prove that in a UFD R an element is prime iff it is irreducible. 8

UNIT – V

9. (a) Let R be a ring with unity.
Prove that an R–module M is cyclic iff $M \simeq \frac{R}{I}$ for some left ideal I of R. 8

(b) Prove that,
Let $M = \sum_{\alpha \in \Lambda} M_\alpha$ be a sum of simple R–submodules M_α. Let K be a submodule of M. Then there exists a subset Λ' of Λ such that $\sum_{\alpha \in \Lambda'} M_\alpha$ is a direct sum, and $M = K \oplus (\oplus \sum_{\alpha \in \Lambda'} M_\alpha)$. 8