AV–1870

M.Sc. (Part—I) Semester—I (C.B.C.S. Scheme) Examination

102 : MATHEMATICS

(Advanced Abstract Algebra)

Time : Three Hours] [Maximum Marks : 80

Note :— Solve any **ONE** question from each Unit.

UNIT—I

1. (a) Prove that, a subgroup H of a group G is normal in G if and only if,

$g^{-1}Hg = H \ \forall \ g \in G$ iff

$g^{-1}hg \in H \ \forall \ h \in H, \ g \in G.$ 8

(b) Let G be a finite group of order p^n, where p is prime and $n > 0$ then prove that :

(i) G has a nontrivial center Z.

(ii) $Z \cap N$ is nontrivial for any nontrivial normal subgroup N of G. 8

2. (c) Let G be a group and let X be a set. Prove that :

(i) If X is a G-set, then the action of G on X induces a homomorphism $\phi : G \to S_X$.

(ii) Any homomorphism $\phi : G \to S_X$ induces an action of G onto X. 8

(d) Let H and K be normal subgroups of a group G and $K \subseteq H$. Prove that $\frac{G}{H} \simeq \frac{G/K}{H/K}$. 8

UNIT—II

3. (a) Let G be a group of order 108. Show that there exist a normal subgroup of order 27 or 9. 8

(b) Let p be a prime dividing $0(G)$ where G is a finite group. Show that :

(i) If K is normal in G and P is a Sylow p-subgroup, then $P \cap K$ is a Sylow p-subgroup of G.

(ii) $\frac{PK}{K}$ is a Sylow p-subgroup of $\frac{G}{K}$.

(iii) Every Sylow p-subgroup of $\frac{G}{K}$ is of the form $\frac{PK}{K}$ where P is a Sylow p-subgroup of G. 8

4. (c) Prove that, a sylow p-subgroup of a finite group G is unique iff it is normal. 8

(d) Define Alternating group A_n. Prove that A_n, $n > 4$ is the only nontrivial normal subgroup of S_n. 8

UNIT—III

5. (a) Let A and B be two ideals of a ring R, then prove that :

(i) $\frac{A+B}{B} \cong \frac{A}{A \cap B}$

(ii) $\frac{Z}{<2>} \cong \frac{5Z}{10Z}$. 8

(b) Let R be a commutative ring. Prove that an ideal P of R is a prime ideal iff for two ideals A, B of R, $AB \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$. 8

6. (c) Prove that. Let $R_1, R_2, \ldots, R_n$ be a family of rings. and let $R = R_1 \times R_2 \times R_3 \times \ldots \times R_n$ be their direct product. Let $R_i^{*} = \{(0, \ldots 0, a_i, 0, \ldots 0)/a_i \in R_i\}$. Then $R = \oplus \sum_{i=1}^{n} R_i^{*}$ is a direct sum of ideals R_i^{*} and $R_i^{*} \simeq R_i$ as rings; on the other hand, if $R = \oplus \sum_{i=1}^{n} A_i$, a direct sum of ideals of R, then $R \simeq A_1 \times A_2 \times \ldots \times A_n$, the direct product of $A_i's$ considered as rings on their own right. 8

(d) Define :

(i) Maximal ideal

(ii) Prime ideal.

In ring R with unity, prove that each maximal ideal is prime. But the converse is in general not true. 8

UNIT—IV

7. (a) Define Unique Factorization Domains and prove that commutative integral domain $R = \{a + b\sqrt{-5} / a, b \in Z\}$ is not unique factorization domain. 8

(b) Define :

(i) Prime Element

(ii) Irreducible Element.

Prove that, in a Principal Ideal Domain (PID) an element is prime if and only if it is irreducible. 8

8. (c) Define :

(i) Euclidean Domain (ED)

(ii) Principal Ideal Domain (PID).

Prove that, every ED is a PID. 8

(d) Prove that if F is a field, then F[x] is a Euclidean Domain. 8

UNIT—V

9. (a) Let R be a ring with unity. Let $Hom_R(R, R)$ denote the ring of endomorphism of R regarded as a right R-module. Then prove that. $R \simeq Hom_R(R, R)$ as rings. 8

(b) Define :

(i) Cyclic Module

(ii) Simple R-module

and let M be a simple R-module then show that $Hom_R(M, M)$ is a division ring. 8

10. (c) If M is an R-module and $x \in M$ then show that the set $K = \{rx + nx / r \in R, n \in Z\}$ is an R-submodule of M containing x. Further, if R has unity, then $K = Rx$. 8

(d) Let A and B be R-submodules of R-module M and N respectively. Then show that $\frac{M \times N}{A \times B} \simeq \frac{M}{A} \times \frac{N}{B}$. 8