(d) Let $f(z) = \sum_{n=0}^{\infty} z^n$ and $g(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$

Show that f and g are direct analytic continuation of each other. 8

AQ – 807

First Semester M. Sc. (Part–I)(C. B. C. S.) Examination

(New Course)

MATHEMATICS

Complex Analysis

P. Pages : 6

Time : Three Hours] [Max. Marks : 80

Note : Solve one question from each unit.

UNIT I

1. (a) Let f be analytic in B (a , R) then prove that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \text{ for } |z-a| < R,$$

where $a_n = \dfrac{f^{(n)}(a)}{n!}$ and this series has radius of convergence $\geq$ R. 8

(b) (i) Prove that If f is bounded entire function then f is constant

(ii) Evaluate $\int_{r} \frac{ez^2}{(z-i)^4} dz, r(t) = 2e^{it}$ $0 \leq t \leq 2\pi$ 8

2. (c) Let $f : G \to C$ be a function suppose that $\overline{B}(a,r) \subset G (r>o)$.
If $r(t)=a+re^{it}$ $0 \le t \le 2\pi$ then prove that,

$$f(z) = \frac{1}{2\pi i}\int_r \frac{f(w)}{w-z}\,dw \text{ for } |z-a|<r$$

Also find the value of $\int_r \frac{dz}{z+\pi i}$, where r is

$|z + 3i| = 1$. 8

(d) (i) Prove that if $P(z)$ is non – constant polynomial function then there is a complex no. 'a' such that $p(a)=0$.

(ii) Prove that zeroes of analytic functions are isolated. Also find the zeroes of $f(z)=z^7+2z^6$ 8

UNIT II

3. (a) Let G be a Region, let f be function from G to $\mathbb{C}$ be a continuous function such that $\int_T f = 0$, for every triangular path in G. Then prove that 'f' is analytic function. 8

(b) (i) Express $f(z) = \frac{1}{2z^2+5z-3}$ as a Taylor's series in the Region $|z| < 1$.

(ii) Obtain Taylors series of the function $f(z) = \sinh z$. Also find radius of convergence. 8

4. (c) State and prove Maximum Modulus theorem. Also show that sin z is unbounded for $z \in \mathbb{C}$. 8

(d) Prove that, a non – constant anayltic function maps open set into open set. 8

UNIT III

5. (a) State and prove Casorati Weierstrass theorem. 8

(b) Expand $\frac{1}{z(z^2-3z+2)}$ in

(i) ann $(0,0,1)$

(ii) ann $(0,1,2)$ and (iii) ann $(0,2,\infty)$. 8

6. (c) State Rouche's Theorem. Also show that 3 zeros of the function $z^4 - 7z - 1$ lie inside the ann $(0,1,2)$. 8

(d) An isolated singularity z = a of the function f (z) is Removable iff $\lim_{z \to a} (z - a) f(z) = 0$. Prove this. 8

UNIT IV

7. (a) Let f be analytic in the Region G except for the isolated singularity $a_1, a_2 \ldots\ldots a_m$. Let r be a closed rectifiable curve not passing through $a_1, a_2 \ldots\ldots a_m$ then prove that

$$\frac{1}{2\pi i} \int_r f(z)\, dz = \sum_{K=1}^{m} n(r, a_K) \operatorname{Res}(f, a_K)$$

Hence find residue of function $e^{2/z}$. Also find the value of integral $\int_r e^{2/z}$ where r is a closed curve about z = 0. 8

(b) Show that $\int_0^{\pi} \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2 - 1}}$, for a > 1 8

8. (c) (i) If $\{ f_n \} \subseteq H(G)$ converges to f in H (G) and each f_n never vanishes on G then prove that either $f \equiv 0$ or f never vanishes on G.

(ii) Evaluate $\int_r \frac{dz}{z^4 + z^3 - 2z^2}$, where r is closed curve | z | = 3. 8

(d) Show that $\int_0^{\infty} \frac{x^{-C}}{1 + x} dx = \frac{\pi}{\sin \pi c}$ $0 < c < 1$ 8

UNIT V

9. (a) Find analytic continuation of

$$\int_0^{\infty} (1 + t) e^{-zt} dt, \text{ for } \operatorname{Re} z < 0.$$ 8

(b) Define Natural boundary.
Explain power series method of analytic continuation. 8

10. (c) Let $r : [a, b] \to \mathbb{C}$ be a path from a to b. Let $\{ (f_t, D_t) : 0 \leq t \leq 1 \}$ and $\{ (g_t, B_t) : 0 \leq t \leq 1 \}$ be analytic continuation along r such that $[f_0]_a = [g_0]_a$ then prove that $[f_1]_b = [g_1]_b$. 8