AU–218

M.Sc. (Part—I) Semester—I (C.B.C.S. Scheme) Examination

103 : MATHEMATICS

(Complex Analysis)

Time : Three Hours] [Maximum Marks : 80

Note :— Solve ONE question from each Unit.

UNIT—I

1. (a) State and prove fundamental theorem of algebra. 2+6

(b) Define zero of an analytic function with multiplicity m, where $m \in \mathbb{N}$. Also show that zeros of analytic function are isolated. 2+6

2. (c) Define analytic function. Let $f : G \to \mathbb{C}$ be a function with $\overline{B}(a, r) \subseteq G, (r > 0)$. If $\gamma(t) = a + re^{it}, 0 \le t \le 2\pi$ then prove that $f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(w)}{(w - z)^{n+1}} dw$ for $|z - a| < r$

and hence evaluate :

$$\int_\gamma z^{-2} e^{iz} dz, \text{ where } \gamma(t) = e^{it}, 0 \le t \le 2\pi.$$ 2+6+2

(d) Let f be analytic in $|z| < 5$ and suppose $|f(z)| \le 10$, for all z in side $|z - 1| = 3$. Then find the bounds of $f^{(3)}(1)$ and $f^{(3)}(0)$. 3+3

UNIT—II

3. (a) Let G be a region and suppose that f is a non-constant analytic function on G. Then show that for any open set U in G, f(U) is open. 8

(b) Define index of a closed curve and evaluate $\int_\gamma \frac{dz}{z^2 + 1}$ where γ is a closed curve not passing through i and –i. 2+6

4. (c) Let G be an open set and let $f : G \to \mathbb{C}$ be differentiable function, then show that f is analytic on G. 8

(d) Find Taylor's series for $f(z) = \sin^3 z$ about $z = 0$ and evaluate :

$$\int_\gamma \frac{dz}{z^2 - 4}, \text{ where } \gamma \text{ is } |z - 1| = 2.$$ 4+4

UNIT—III

5. (a) Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent's series for the following region :

$1 < |z| < 3.$ 4

(b) (i) Define essential singularity

(ii) State and prove Casorti-Wierstrass theorem. 2+2+8

6. (c) State and prove Rouche's theorem. 2+6

(d) Let f be analytic in the annulus ann(a; R_1, R_2) then prove that $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$, where the convergence is absolute and uniform over ann $(a; r_1, r_2)^-$, if $R_1 < r_1 < r_2 < R_2$. Also show that the coefficients a_n are given by formula $a_n = \dfrac{1}{2\pi i}\int_\gamma \dfrac{f(z)}{(z-a)^{n+1}}\,dz$, where γ is a circle $|z-a| = r$, for any r, $R_1 < r < R_2$. 8

UNIT—IV

7. (a) State and prove Hurwitz's theorem. 2+6

(b) Evaluate $\int_{-\infty}^{\infty} \dfrac{x^2}{1+x^4}\,dx$. 8

8. (c) Evaluate $\int_0^{\pi} \dfrac{d\theta}{1+3\cos^2\theta}$. 8

(d) State and prove Hadamard's three circle theorem. 2+6

UNIT—V

9. (a) State and prove Schwartz Reflection principle. 2+6

(b) Let $r : [0, 1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, D_t) : 0 \le t \le 1\}$ and $\{(g_t, B_t) : 0 \le t \le 1\}$ be analytic continuous along r such that $[f_0]_a = [g_0]_a$. Then prove that $[f_1]_b = [g_1]_b$. 8

10. (c) Define natural boundary. Explain power series method of analytic continuation. 2+6

(d) Let f be an entire function and let $\{a_n\}$ be the non-zero zeros of f repeated according to multiplicity; suppose f has a zero at z = 0 of order $m \ge 0$ (a zero of order m = 0 at z = 0 means $f(0) \neq 0$). Then prove that there is an entire function g and a sequence of integers $\{p_n\}$ such that

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$ 8