AV–1871

M.Sc. (Part—I) Semester—I (C.B.C.S. Scheme) Examination

103 : MATHEMATICS

(Complex Analysis)

Time : Three Hours] [Maximum Marks : 80

N.B. :— Solve **ONE** question from each unit.

UNIT—I

1. (a) Let $f : G \to \mathbb{C}$ be a function with $\overline{B}(a, r) \subseteq G\ (r > 0)$. If $\gamma(t) = a + r e^{it}$, $0 \le t \le 2\pi$ then prove that, $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \dfrac{f(w)}{w - z}\, dw, |z - a| < r$ and hence evaluate $\displaystyle\int_\gamma \dfrac{dz}{z + \pi i}$, where γ is $|z + 3i| = 1$. 6+2

   (b) State and prove Cauchy's theorem. 2+6

2. (c) Define :

   (i) Zero of an analytic function with multiplicity m

   (ii) Entire function. 2+2

   (d) (i) If f is bounded entire function then show that f is constant. 4

   (ii) State and prove fundamental theorem of algebra. 2+6

UNIT—II

3. (a) Define an index of a closed curve and show that it is an integer. 2+6

   (b) (i) Evaluate $\displaystyle\int_\gamma \dfrac{dz}{z^2 + 1}$, where γ is a closed curve not passing through i and –i.

   (ii) Evaluate $\displaystyle\int_\gamma \dfrac{dz}{z^2 - 4}$, where γ is $|z - 1| = 2$. 4+4

4. (c) State and prove maximum modulus theorem. 2+6

   (d) Let G be a region and suppose that f is nonconstant analytic function on G. Then prove that for any open set U in G, f(U) is open. 2+6

UNIT—III

5. (a) State and prove Casorati Weistrass theorem. 2+6

   (b) Prove that an isolated singularity z = a of a function f(z) is removable iff $\lim_{z \to a} (z - a)\, f(z) = 0$. 8

6. (c) State Rouche's theorem and show that all roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$. 2+6

   (d) Define :

   (i) Essential singularity

   (ii) Isolated singularity

   (iii) Pole of order m

   (iv) Removable singularity. 2+2+2+2

UNIT—IV

7. (a) State and prove Cauchy Residue Theorem. 2+6

(b) Show that $\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\, dx = \frac{\pi}{\sqrt{2}}$. 8

8. (c) State and prove Hadamard's three circle theorem. 2+6

(d) Evaluate $\int_0^{\pi} \frac{d\theta}{a + \cos\theta}$ where $a > 1$ and hence show that $\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{3/2}}$. 6+2

UNIT—V

9. (a) State and prove Weistrass Factorization theorem. 2+6

(b) Let $f(z) = \sum_{n=0}^{\infty} z^n$ and $g(z) = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{1+z}{2}\right)^n$, show that f and g are direct analytic continuation of each other. 8

10. (c) Show that unit circle is a natural boundary of the function $1 + \sum_{n=0}^{\infty} z^{2^n}$. 8

(d) Let $\gamma : [0, 1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, D_t) : 0 \le t \le 1\}$ and $\{(g_t, B_t) : 0 \le t \le 1\}$ be analytic continuous along γ such that $[f_0]_a = [g_0]_a$. Then prove that $[f_1]_b = [g_1]_b$. 8