AQ – 799

First Semester M. Sc. (Part–I)(C. B. C. S. Pattern) Examination

(Old Course)

MATHEMATICS (102)

Advanced Abstract Algebra – I

P. Pages : 7

Time : Three Hours] [Max. Marks : 80

Note : Solve 'one' question from each unit.

UNIT – I

1. (a) Define :—

   (i) Normal series.

   (ii) Composition series.

   Prove that a group G is solvable if and only if G has normal series with abelian factors. Further a finite group is solvable if and only if it's composition factors are cyclic groups of prime orders. 1 + 1 + 6 = 8

   (b) Let G be a finite group of order P^n, where P is prime and $n > 0$, then prove that

   (i) G has a non – trivial centre Z.

(ii) $Z \cap N$ is nontrivial for any non – trivial normal subgroup N of G.

(iii) Prove that :

If H is a proper subgroup of G, then H is properly contained in N (H) ; hence if H is a subgroup of order P^{n-1}, then $H \triangle G$ 8

2. (c) Prove that, every nilpotent group is solvable, and if G is nilpotent group, then prove that every subgroup of G and every homomorphic image of G are nilpotent. 8

(d) Define, orbit of x in G.

Let G be a group acting on set X. then the set of all orbits in X under G is partition of X. For any $x \in X$ there is a bijection $G_x \to G / G_x$ and hence $|G_x| = [G : G_x]$ therefore, if X is a finite set,

$$|X| = \sum_{x \in C} [G : Gx]$$

where C is a subset of X containing exactly one element from each orbit. Prove this.

1 + 7 = 8

(ii) Let R be a noetherian ring, then the sum of nilpotent ideals in R is a nilpotent ideal. Prove this. 3

10. (c) State and prove Hilbert – Basis theorem.

1 + 7 = 8

(d) Define :—

(i) Noetherian R – module M.

(ii) Left artinian ring.

Also Prove that, if J is a nil left ideal in an artinian ring R, then J is nilpotent.

1+1+6=8

www.sgbauonline.com

(2) Prove that, every Euclidean domain is a PID. 3

(b) Prove that every PID is a Unique Factorization Domain (UFD) but a UFD is not necessarily a PID. 8

8. (c) Define primitive polynomial, also state and prove Gauss Lemma. 1 + 1 + 6 = 8

(d) Let R be a unique factorization domain (UFD), then prove that the polynomial ring R [x] over R is also a UFD. 8

UNIT – V

9. (a) Define :—

(i) Finitely generated R – module M.

(ii) Free Module.

Let M be a finitely generated free module over a commutative ring R, then prove that all bases of M have the same number of elements. 1 + 1 + 6 = 8

(b) (i) Let M be a simple R – module, then prove that Hom_R (M, M) is a division ring. 5

UNIT – II

3. (a) Prove that :—

(i) If a group of order P^n contains exactly one subgroup each of orders P , P^2 ..... P^{n-1} then it is cyclic.

(ii) There are no simple subgroups of orders 63 and 56. Prove this. 4 + 4 = 8

(b) Let G be a group of order pq, where p and q are prime numbers such that p > q and qX (p – 1). Then G is cyclic. Prove this. 8

4. (c) Define invarients of A.

Let A be a finite abelian group of order $P_1^{e_1}$, ... $P_k^{e_k}$, P_i distinct primes. $e_i > 0$ then prove that,

A = S (p_1) ⊕ ..... ⊕ S (P_K). where,

| S (p_i) | = $P_i^{e_i}$ and this decomposition of A is unique, that is if,

A = H_1 ⊕ ..... ⊕ H_K , where | Hi | = $P_i^{e_i}$ then H_i = S (P_i). 8

(d) State and prove first sylow theorem. 1 + 7 = 8

www.sgbauonline.com

UNIT – III

5. (a) Define :—

(i) Maximal ideal.

(ii) Simple ring.

Also, prove that in a non – zero commutative ring with unity, an ideal M is maximal if and only if R/M is a field. 1 + 1 + 6 = 8

(b) Let $f : R \to S$ be a homomorphism of a ring R on to a ring S, and let N = K erf. then the mapping $F : A \to f(A)$ defines a 1 – 1 correspondence from the set of all ideals (right ideals, left ideals) in R that contain N onto the set of all ideals (right ideals, left ideals) in S. It preserves ordering in the sense that $A \subsetneq B$ iff $f(A) \subsetneq f(B)$. Prove this. 8

6. (c) (i) Let f be a homomorphism of ring R into a ring S with Kernel N, then prove that $R/N \simeq \text{Im} f$. 4

(ii) Let R be a commutative ring with unity in which each ideal is prime, then prove that R is a field. 4

(d) Define :—

(i) Sum of ideal,

(ii) Direct sum of ideal.

Let $A_1, A_2 \ldots\ldots A_n$ be right (or left) ideals in ring R, then prove that the following are equivalent.

(i) $A = \sum_{i=1}^{n} A_i$, is a direct sum.

(ii) If $0 = \sum_{i=1}^{n} a_i$, $a_i \in A_i$, then $a_i = 0$ $i = 1, 2 \ldots\ldots n$.

(iii) $A_i \cap \sum_{j=1, j \neq i}^{n} A_j = 0$, $i = 1, 2 \ldots\ldots n$.

1 + 1 + 6 = 8

UNIT – IV

7. (a) (1) Define :—

(i) Irreducible element

(ii) Prime element.

Prove that an irreducible element in a commutative principle ideal domain (PID) is always prime. 1 + 1 + 3 = 5