6. (c) Prove that the product of two convergent series converges if atleast one of the series converges absolutely. 8

(d) State and prove Abel's theorem. 8

UNIT IV

7. (a) Suppose X is a vector space and dim X = n Then prove the following.

(i) A set E of n vectors in X spans X if and only if E is independent. 8

(ii) X has a basis and every basis consists of n vectors. 8

(b) Suppose E is an open set in R^n, f maps E into R^m, f is differentiable at $x_0 \in E$, g maps an open set containing f(E) into R^k, and g is differentiable at $f(x_0)$. Then prove that the mapping F of E into R^k defined by F(x) = g (f(x)) is differentiable at x_0 and

$F'(x_0) = g'(f(x_0))\ f'(x_0)$. 8

8. (c) Suppose f maps an open set $E \subset R^n$ into R^m Then prove that $f \in C'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$ and $1 \leq j \leq n$. 8

First Semester M. Sc. (Part – I) (CBCS.) Examination

(Old)

MATHEMATICS

Paper – I (MTH - I)

(Real Analysis)

P. Pages : 6

Note : Attempt five questions in all selecting one question from each unit.

UNIT I

1. (a) (i) If f is monotonic on [a, b] and if α is continuous on [a, b], then prove that $f \in \mathscr{R}(\alpha)$. 4

(ii) If $f \in \mathscr{R}(\alpha)$ on [a, b] and if $|f(x)| \leq m$ on [a, b] then prove that $\left|\int_a^b f\,d\alpha\right| \leq m\ \{\alpha(b) - \alpha(a)\}$. 4

(b) Define two functions β_1, β_2 as follows :

$\beta_j(x) = 0$ if $x < 0$,

$\beta_j(x) = 1$ if $x > 0$ for j = 1, 2 and

$\beta_1(0) = 0$, $\beta_2(0) = 1$. Let f be a bounded function on $[-1, 1]$.

(i) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then $\int f d\beta_1 = f(0)$.

(ii) State a similar result for β_2. 8

2. (c) Define a rectifiable curve. Given a curve γ whose derivative γ' is continuous on $[a, b]$ then show that γ is a rectifiable curve and has length

$$\int_a^b |\gamma'(t)| dt.$$ 10

(d) State and prove fundamental theorem of integral calculus for vector valued functions. 6

UNIT II

3. (a) State and prove Cauchy's Criterion for uniform convergence of a sequence of a functions $\{fn(x)\}$. 6

(b) Suppose $\{fn(x)\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{fn(xo)\}$ converges for some point x_0 on $[a, b]$. If $\{fn'(x)\}$ converges uniformly on $[a, b]$ then prove that $\{fn\}$ converges uniformly on $[a, b]$, to a function f, and

$$f'(x) = \lim_{n\to\infty} f_n'(x) \quad (a \leq x \leq b).$$ 10

4. (c) State and prove the Stone-Weierstrass theorem for a sequence of polynomials P_n. 10

(d) Show that the limit of the integral need not be equal to the integral of the limit in case of

$$fn(x) = n^2x(1 - x^2)^n, \ 0 \leq x \leq 1, \ n = 1, 2, 3 \text{----}$$ 6

UNIT III

5. (a) Let Σa_n be a series of real numbers which converges, but not absolutely. Suppose $-\infty \leq \alpha \leq \beta \leq \infty$. Then prove that there exists a rearrangement $\Sigma a_n'$ with partial sums S_n' such that $\lim_{n\to\infty} \inf S_n' = \alpha$ and $\lim_{n\to\infty} \sup S_n' = \beta$. 8

(b) Given a double sequence $\{a_{ij}\}$, $i = 1, 2, 3$----
$j = 1, 2, 3$----

Suppose that $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ $(i = 1, 2, 3\text{---})$ and Σbi converges. Then prove that

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$ 8

(d) Suppose $\overline{f}: E \longrightarrow IR^m$, is differentiable at $\overline{x} \in ECIR^n$. Show that the differential of $\overline{f}$ at $\overline{x}$ is unique. Further find the differential of $\overline{f}$ if $\overline{f} : IR^n \longrightarrow IR^m$ is a linear transformation. 8

UNIT V

9. (a) Suppose f is a C′– mapping of an open set $E C R^n$ into R^n, $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. Then prove that there exist open sets U and V in R^n such that $a \in U$, $b \in V$, f is one–to–one on U and $f(U) = V$. 12

(b) Find the maxima and minima of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$. 4

10. (c) Let f be a C′– mapping of an open set $E C R^{n+m}$ into R^n, such that $f(a, b) = 0$ for some point $(a, b) \in E$. Put $A = f'(a, b)$ and assume that Ax is invertible. Then show that there exists open sets $U C R^{n+m}$ and $W C R^m$ with $(a, b) \in U$ and $b \in W$ having the following property :
To every $y \in W$ corresponds a unique x, such that $(x, y) \in U$ and $f(x, y) = 0$. 10

(d) If $x + y + z = u$; $y + z = uv$; $z = uvw$ then show that

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2v.$$ 6