AU–216

M.Sc. Part—I) Semester—I (C.B.C.S. Scheme) Examination

MATHEMATICS

(Real Analysis)

Paper—101

Time : Three Hours] [Maximum Marks : 80

Note :— Solve any **ONE** question from each Unit.

UNIT—I

1. (a) The function α increases on [a, b] and is continuous at x_0, $a \le x_0 \le b$. If $f(x_0) = 1$ and $f(x) = 0,\ x \ne x_0$. Then show that $f \in R(\alpha)$ and $\int_a^b f\, d\alpha = 0$. 8

   (b) State and prove fundamental theorem of integral calculus for vector valued function. 8

2. (c) Define :

   (i) Simple closed curve

   (ii) Rectifiable curve.

   If the curve γ whose derivative is continuous on [a, b] then show that γ is rectifiable and has length $\int_a^b |\gamma'(t)|\, dt$. 8

   (d) Let $C_n \ge 0$ for n = 1, 2, 3, ..., $\sum_{n=1}^{\infty} C_n$ converges and $(s_n)_{n=1}^{\infty}$ be a sequence of distinct points in (a, b) and $\alpha(x) = \sum_{n=1}^{\infty} C_n\, I(x - S_n)$. If f is a continuous real valued function defined on [a, b] then show that :

   $$\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} C_n\, f(S_n).$$ 8

UNIT—II

3. (a) Suppose K is compact, and :

   (i) $\{f_n\}$ is a sequence of continuous function on K.

   (ii) $\{f_n\}$ converges pointwise to a continuous function f on K.

   (iii) $f_n(x) \geq f_{n+1}(x) \ \forall \ x \in K, n = 1, 2, 3, \ldots$

   Then prove that $f_n \to f$ uniformly on K. 8

   (b) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n + x^2}$ is uniformly convergent but not absolutely for all real values of x. 8

4. (c) Show that the sequence $\{f_n\} = \tan^{-1} nx$, $n \geq 0$ is uniformly convergent on [a, b], a > 0 but pointwise convergent in [0, b]. 8

   (d) Show that there exists a real continuous function on the real line which is nowhere differentiable. 8

UNIT—III

5. (a) State and prove Abel's theorem. 8

   (b) If two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge on the same interval $(-R, R)$ $(R > 0)$ to some function f(x), then show that $a_n = b_n$ for n = 0, 1, 2, 3, ... 8

6. (c) Suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ converges to f(x) in (−1, 1) and $\lim_{n \to \infty} n a_n = 0$. If $\lim_{x \to 1^-} f(x) = A$, then show that $\sum_{n=0}^{\infty} a_n$ converges to A. 8

   (d) Find the radius of convergence of following power series :

   (i) $\sum \frac{n x^n}{(n+1)^2}$

   (ii) $\sum \frac{2^n x^n}{n!}$

   (iii) $\sum (-1)^n \frac{x^{2n+1}}{(2n+1)!}$. 8

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UNIT—IV

7. (a) Prove that :

   (i) If $A \in L(R^n, R^m)$ then $\|A\| < \infty$ and a is a uniformly continuous mapping of R^n into R^m.

   (ii) If $A, B \in L(R^n, R^m)$ and C is scalar, then $\|A + B\| \leq \|A\| + \|B\|$, $\|C A\| = |C| \cdot \|A\|$ with the distance between A and B defined as $\|A - B\|$, $L(R^n, R^m)$ is metric space. 8

   (b) Show that the partial derivatives f_{xy} and f_{yx} of the function

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, \quad (x, y) \neq (0, 0)$$
$$= 0 \qquad , \quad (x, y) = (0, 0)$$

   are not equal at origin. 8

8. (c) State and prove Taylor's theorem. 8

   (d) Show that the function :

$$f(x, y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2} & , (x, y) \neq (0, 0) \\ 0 & , (x, y) = (0, 0) \end{cases}$$

   is continuous at (0, 0) but not differentiable at (0, 0). 8

UNIT—V

9. (a) Suppose f is a C'-mapping of an open set $E \subset R^n$ into R^n, $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$. Then show that :

   (i) There exist open sets U and V in R^n such that $a \in U$, $b \in V$, f is one-one on U and $f(U) = V$

   (ii) If g is the inverse of f (which exists by (i)), defined in V by $g(f(x)) = x$, $x \in U$ then $g \in C'(V)$. 12

   (b) Show that the function $f(x, y) = 2x^4 - 3x^2y + y^2$ has neither a maximum nor a minimum at (0, 0), where $f_{xx}f_{yy} - (f_{xy})^2 = 0$. 4

10. (c) Find the maximum and minimum value of the function :

$$f(x, y) = 21x - 12x^2 - 2y^2 + x^3 + xy^2.$$ 8

   (d) The roots of the equation in λ :

   $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ are u, v, w. Prove that :

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(y - z)(z - x)(x - y)}{(v - w)(w - u)(u - v)}.$$ 8