AV–1869

M.Sc. (Part—I) Semester—I (C.B.C.S. Scheme) Examination
101 :MATHEMATICS
(Real Analysis)

Time : Three Hours] [Maximum Marks : 80

Note :— Solve any **ONE** question from each unit.

UNIT—I

1. (a) Prove that, a necessary and sufficient condition for $f \in R(\alpha)$ on I is that for every $\in > 0$, $\exists$ a partition P on I such that $U(P) - L(P) < \in$. 8

(b) If $f \in R(\alpha)$ and $g \in R(\alpha)$ on $[a, b]$ then prove that :

(i) $f + g \in R(\alpha)$

(ii) $f \cdot g \in R(\alpha)$. 8

2. (c) Let $f \in R(\alpha)$ on $[a, b]$, $m \le f \le M$. Let the function F be continuous on $[m, M]$ and $h(x) = F(f(x))$ on $[a, b]$. Then prove that $h \in R(\alpha)$ on $[a, b]$. 8

(d) Define :

(i) Simple closed curve

(ii) Closed curve

If $f(x) = \begin{cases} 0, & x \text{ is irrational} \\ 1, & x \text{ is rational} \end{cases}$

then show that $f \notin R$ on $[a, b]$ for any $a < b$. 8

UNIT—II

3. (a) State and prove Weierstrass approximation theorem. 10

(b) Show that the series $\sum \dfrac{x}{(nx+1)\{(n-1)x+1\}}$ is uniformly convergent on any interval, $[a, b]$, $0 < a < b$, but not uniformly on $[0, b]$. 6

4. (c) For $n = 1, 2, ...$ and x real, $f_n(x) = \dfrac{x}{1+nx^2}$. Show that $\{f_n\}$ converges uniformly to a function f and that the equation $f'(x) = \lim\limits_{n \to \infty} f_n'(x)$ is correct if $x \ne 0$ but false if $x = 0$.

8

(d) Show that $\lim\limits_{m \to \infty} \lim\limits_{n \to \infty} [\cos m!\,\pi x]^{2n}$ is everywhere discontinuous function which is not Riemann integrable. 8

UNIT—III

5. (a) State and prove Taylor's theorem. 8

(b) If R is the radius of convergence of the power series $\sum\limits_{n=0}^{\infty} a_n x^n$ then prove that, radius of convergence of the power series $\sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ is also R. 8

6. (c) Show that :

(i) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \ -1 \le x \le 1$

(ii) $\frac{1}{2}[\log(1+x)]^2 = \frac{x^2}{2} - \frac{x^3}{3}\left(1+\frac{1}{2}\right) + \frac{x^4}{4}\left(1+1\frac{1}{2}+\frac{1}{3}\right) - \ldots, \ldots, -1 \le x \le 1$ 8

(d) State and prove uniqueness theorem for the power series. 8

UNIT—IV

7. (a) Let Ω be the set of all invertible linear operators on R^n :

(i) If $A \in \Omega$, $\|A^{-1}\| = \frac{1}{\alpha}$, $B \in L(R^n)$ and $\|B - A\| < \alpha$, then prove that $B \in \Omega$.

(ii) Prove that Ω is an open subset of $L(R^n)$ and the mapping $A \to A^{-1}$ is continuous on Ω. 8

(b) Prove that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at the point (0, 0), but that f_x and f_y both exist at the origin and have the value zero. Hence deduce that these two partial derivatives are continuous except at the origin. 8

8. (c) Suppose $\bar{f}$ map an open set $E \subset R^n$ into R^m. Then $\bar{f} \in \mathscr{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \le i \le m$, $1 \le j \le n$. 8

(d) Show that the function f where :

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}}, & \text{if } x^2 + y^2 \ne 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

is continuous, possesses partial derivatives but is not differentiable at the origin. 8

UNIT—V

9. (a) State and prove Implicit function theorem. 12

(b) Find the maximum and minimum value of the function :

$f(x, y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4.$ 4

10. (c) Suppose f is a $\mathscr{C}'$-mapping of an open set $E \subset R^n$ into R^n, $f'(a)$ is invertible for some $a \in E$, and $b = f(a)$. Then prove that there exist an open sets U and V in R^n such that $a \in U$, $b \in V$, f is one to one on U and $f(U) = V$. 8

(d) Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the conditions $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ and $z = x + y$. 8