M.Sc. Part–I (Semester–II) (CBCS Scheme) Examination (Old)

MATHEMATICS

(Advanced Abstract Algebra–II)

Paper—VII-202

Time—Three Hours] [Maximum Marks—80

Note :— Solve **FIVE** questions, selecting **ONE** from each Unit.

UNIT—I

1. (a) Let $f(x) = a_0 + a_1x + ... + a_nx^n \in Z[x]$, $n \geq 1$. Prove that, if there is a prime p such that $p^2 \nmid a_0$, $p|a_0$, $p|a_1$, ..., $p|a_{n-1}$, $p \nmid a_n$, then f(x) is irreducible over **Q**. 8

(b) Let p(x) be an irreducible polynomial in F[x] and let u be a root of p(x) in an extension E of F, then prove that :

(i) F(u), the subfield of E generated by F and u is the set

$F[u] = \{b_0 + b_1u + ... + b_mu^m \in E / b_0 + b_1x + ... + b_mx^m \in F[x]\}$

(ii) If the degree of p(x) is n, the set $(1, u, ... u^{n-1})$ forms a basis of F(u) over F and [F(u) : F] =

2\. (c) Define the splitting field of $f(x)$ over field F. Let K be a splitting field of the polynomial $f(x) \in F[x]$ over a field F. If E is another splitting field of $f(x)$ over F then prove that there exists an isomorphism $\sigma : E \to K$ that is identity on F. 8

(d) Let E be an algebraic extension of a field F contained in an algebraic closure $\overline{F}$ of F. Then prove that the following conditions are equivalent :

(i) Every irreducible polynomial in F[x] that has a root in E splits into linear factors in E

(ii) E is the splitting field of a family of polynomials in F[x]

(iii) Every embedding σ of E in $\overline{F}$ that keeps each element of F fixed maps E onto E. 8

UNIT—II

3\. (a) Define the prime field and prove that the prime field of a field F is either isomorphic to $\mathbf{Q}$ or to $Z/(p)$ where p is a prime. 8

(b) State and prove the fundamental theorem of Algebra. 8

4\. (c) Prove that any finite field F with p^n elements is the splitting field of $x^{p^n} - x \in F_p[x]$. Consequently any two finite fields with p^n elements are isomorphic. 8

UNIT–V

9\. (a) Let M be a finitely generated module over a principal ideal domain R, then prove that :

$$M = F \oplus \text{Tor } M$$

where (i) $F \simeq R^s$ for some nonnegative integer s

(ii) $\text{Tor } M \simeq R/Ra_1 \oplus \dots \oplus R/Ra_r$, where a_i are nonzero nonunit elements in R such that $a_1 \mid a_2 \mid \dots \mid a_r$. 8

(b) Reduce the matrix $A = \begin{bmatrix} -3 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & -3 & -2 \end{bmatrix}$ to rational canonical form. 8

10\. (c) Let W be a subspace of V. Let $T \in \text{Hom}_F(V, V)$ such that $TW \subseteq W$. Then prove that W is a T-cyclic subspace if and only if there exists an element $w \in W$ such that $\{w, T_w, \dots, T_w^{k-1}$ is a basis of W for some $k \geq 1$. 8

(d) Find the invariant factors, elementary divisors and the Jordan canonical form of the matrix :

$$\begin{bmatrix} 0 & 4 & 2 \\ -3 & 8 & 3 \\ 4 & -8 & -2 \end{bmatrix}.$$ 8

(d) Let H be a finite subgroup of the group of automorphisms of a field E. Then prove that $[E : E_H] = |H|$ where $E_H = \{x \in E / \sigma_{(x)} = x \text{ for all } \sigma \in H\}$. 8

UNIT—III

5. (a) Define the n^{th} cyclotomic polynomial $\Phi_n(x)$. Hence find $\Phi_6(x)$. 6

(b) Prove that $\Phi_n(x) = \prod_w (x - w)$ is an irreducible polynomial of degree $\phi(n)$ in $Z[x]$, where w is primitive n^{th} root in $\mathbb{C}$. 10

6. (c) Prove that a polynomial $f(x) \in F[x]$ is solvable by radicals over F if its splitting field E over F has solvable Galois group G(E/F). 8

(d) Show that the polynomial $x^7 - 10x^5 + 15x + 5$ is not solvable by radicals over $\mathbf{Q}$. 8

UNIT–IV

7. (a) Let R be a PID and let F be a free R-module with a basis consisting of n elements. Then prove that any submodule K of F is also free with a basis consisting of m elements such that $m \le n$. 8

(b) Obtain the Smith normal form and rank for the following matrix over the PID $\mathbf{Q}[x]$:

$$\begin{bmatrix} -x-3 & 2 & 0 \\ 1 & -x & 1 \\ 1 & -3 & -x-2 \end{bmatrix}.$$ 8

8. (c) Prove that, if A is an $m \times n$ matrix over a principal ideal domain R, then A is equivalent to a matrix that has the diagonal form

$$\begin{bmatrix} a_1 & & & & & & & & \\ & a_2 & & & & & & & \\ & & \cdot & & & & & & \\ & & & \cdot & & & & & \\ & & & & \cdot & & & & \\ & & & & & a_r & & & \\ & & & & & & 0 & & \\ & & & & & & & \cdot & \\ & & & & & & & & \cdot \\ & & & & & & & & & \cdot \\ & & & & & & & & & & 0 \end{bmatrix},$$

where the $a_i \neq 0$ and $a_1 \mid a_2 \mid \dots \mid a_r$. 8

(d) Find the invariant factors and rank of the matrix

$$\begin{bmatrix} -x & 4 & -2 \\ -3 & 8-x & 3 \\ 4 & -8 & -2-x \end{bmatrix}$$ over the ring $\mathbf{Q}[x]$. 8