8. (c) Prove that the multiplicative group of nonzero elements of a finite field is cyclic. 8

(d) If E is a finite separable extension of a field F, then prove that E is a simple extension of F. 8

UNIT-V

9. (a) Find the Galois groups G(K/Q) of the following extensions K of Q :

(i) $K = Q\left(\sqrt{3}, \sqrt{5}\right)$.

(ii) $K = Q(\alpha)$, where $\alpha = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$

(iii) K is the splitting field of $x^4 - 3x^2 + 4 \in Q[x]$.

2+3+3=8

(b) Prove that $f(x) \in F[x]$ is solvable by radicals over F iff its splitting field E over F has solvable Galois group G(E/F). 8

10. (c) Let E be the splitting field of $x^n - a \in F[x]$. Then prove that G(E/F) is solvable group. 8

(d) Define Cyclotomic polynomial. Prove that the cyclotomic polynomials $\phi_n(x)$ are irreducible over Z. 2+6=8

AQ–883

M.Sc. (Part-I) Semester-II (CBCS Scheme) Examination

MATHEMATICS (New) 202

(Advanced Linear Algebra and Field Theory)

Time—Three Hours] [Maximum Marks—80

Note :— Solve any **ONE** question from each unit.

UNIT-I

1. (a) Prove that :

(i) The characteristic roots of a Hermitian matrix are real.

(ii) The characteristic roots of a unitary matrix are of unit modulus. 8

(b) Determine the eigen values and the corresponding eigen vectors of the matrix :

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ 8

2. (c) Prove that the eigen vectors corresponding to distinct eigen values of a matrix are linearly independent. 8

(d) Let A be a 7×7 matrix with minimal polynomial $M(x) = (x^2 - 2x + 5)(x - 3)^3$. Find all possible rational canonical forms of A. 8

UNIT-II

3. (a) Find canonical form of the matrix :

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

Also find rank and signature of A. 8

(b) Prove that the ranges of values of two congruent quadratic forms are the same. 8

4. (c) Prove that two real quadratic forms in n variables are real equivalent iff they have the same rank and index (or signature). 8

(d) Reduce the following quadratic form to canonical form and find its rank and signature

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -2 & -2 \\ 3 & -2 & 3 \end{bmatrix}$$ 8

UNIT-III

5. (a) Let $F \subseteq E \subseteq K$ be fields. If $[K : E] < \infty$ and $[E : F] < \infty$ then prove that :

(i) $[K : F] < \infty$

(ii) $[K : F] = [K : E][E : F]$. 8

(b) Show that :

(i) $x^3 - x - 1 \in Q[x]$

(ii) $x^4 + 8 \in Q[x]$

are irreducible over Q. 4+4=8

6. (c) Let P(x) be an irreducible polynomial in F[x]. Then prove that there exists an extension E of F in which p(x) has a root. 8

(d) Let F be a field. Then prove that there exists an algebraically closed field K containing F as a sub field. 8

UNIT-IV

7. (a) Prove that any finite field F with p^n elements is the splitting field of $x^{p^n} - x \in F_p[x]$. Consequently, any two finite fields with p^n elements are isomorphic. 8

(b) If the multiplicative group F^* of nonzero elements of a field F is cyclic, then prove that F is finite. 8