AQ–877

M.A./M.Sc. Part–I (Semester–II) (CBCS Scheme) Examination (Old)

MATHEMATICS

(Complex Analysis—II)

Paper—III (203)

Time—Three Hours] [Maximum Marks—80

Note :— Solve **ONE** question from each Unit.

UNIT—I

1. (a) For Re z > 1, prove that :

$$\xi(z)\Gamma z = \int_0^\infty (e^t - 1)^{-1} t^{z-1} dt.$$ 8

(b) Write the statement of Mittag Leffler's theorem and apply it to prove,

$$\cot z = \frac{1}{z} + \sum_{n=-\infty}^{\infty} \left(\frac{1}{z - n\pi} + \frac{1}{n\pi} \right).$$ 8

2. (c) Write the Gauss's formula for Gamma z and derive the functional equation. What are the singular points of Γz, write their nature and find the residue of Γz at those points. 8

(d) Prove that :

(i) $\left\{\left(1+\frac{z}{n}\right)^n\right\}$ converges to e^z in H(C)

(ii) If $t \geq 0$ then $\left(1-\frac{t}{n}\right)^n \leq e^{-t}$ for all $n \geq t$. 8

UNIT—II

3. (a) **State and prove the uniqueness of direct analytic continuation. Also define analytic continuation along a curve.** 8

(b) Show that the power series $z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+\ldots..$ can be analytically continued to a wider region by means of the series,

$$\log 2-\frac{(1-z)}{2}-\frac{1}{2}\left(\frac{1-z}{2}\right)^2-\frac{1}{3}\left(\frac{1-z}{z}\right)^3 \ldots\ldots$$ 8

4. (c) (i) Define Germ of a function at a point.

(ii) Show that if f is analytic in a domain D and let f(z) vanishes over a domain D_0 which is a part of D. Then f(z) vanishes at each point of the whole domain D. 8

10. (c) Let f be analytic function on the region containing the closure of a disk $D = \{z : |z| < 1\}$ and satisfying $f(0) = 0$, $f'(0) = 1$. Then prove that there is a disk $S \subset D$ on which f is one-one and such that f(s) contains a disk of radius $\frac{1}{72}$. 10

(d) Let f be analytic function on a disk B(a, r) such that $|f'(z) - f'(a)| < |f'(a)|$ for all z in B(a, r); $z \neq a$; then prove that f is one-one. 6

(d) (i) Show that $f(z) = \int_0^{\infty} (1 + t)e^{-zt}\, dt$ converges if Re $z > 0$ also,

(ii) Show that $g(z) = \frac{z+1}{z^2}$ is the analytic continuation of the above f(z) into the left half plane Re $z < 0$. 8

UNIT—III

5. (a) Let (f, D) be a function element which admits unrestricted analytic continuation in a simply connected region G. Then prove that there is an analytic function $F : G \to \mathbb{C}$ such that $F(z) = f(z)$ for all z in D. 8

(b) State and prove the theorem to show that Dirichlet's problem can be solved for the unit disk. 8

6. (c) State and prove 'Schwartz's reflection principle'. 8

(d) (i) Define Poisson's kernel and Green's function.

(ii) Prove that if $a \in \mathbb{C}$, $\rho > 0$ and suppose h is a continuous real valued function on $\{z : |z - a| = \rho\}$ then there is a unique continuous function $w : \overline{B}(a, \rho) \to R$ such that w is harmonic on $B(a, \rho)$ and $w(z) = h(z)$ for $|z - a| = \rho$. 8

UNIT—IV

7. (a) State and prove Jensen's formula. 8

(b) Using Hadamard's theorem prove the following :

(i) Let f be an entire function of finite order, then f assumes each complex number with one possible exception

(ii) Let f be an entire function of finite order λ, where λ is not an integer, then f has infinitely many zeros. 8

8. (c) Let f be an entire function of order λ and let $M(r) = \max.\{|f(z)| : |z| = r\}$ then prove that $\lambda = \lim_{r \to \infty} \sup\left[\frac{\log \log M(r)}{\log r}\right]$. Also find the order of the entire function $f(z) = e^{z^n}$, for $n > 0$. 8

(d) Define Genus of an entire function and prove that if f is an entire function of finite genus μ then f is of finite order $\lambda \leq \mu + 1$. 8

UNIT—V

9. (a) Let f be analytic in $D = \{z : |z| < 1\}$. Suppose that $f(0) = 0$, $f'(0) = 1$ and $|f(z)| \leq M$ for all z in D then prove that $M \geq 1$ and $f(D) \supset B\left(0, \frac{1}{6M}\right)$. 8

(b) State and prove 'Little Picard's theorem'. 8