**M.Sc. Part–I (Semester–II) (CBCS Scheme) Examination (Old)**

**MATHEMATICS (201)**

**(Measure and Integration Theory)**

Time—Three Hours] [Maximum Marks—80

**Note :—** Solve **ONE** question from each Unit.

## UNIT—I

1. (a) Show that for any set A, $m^*(A) = m^*(A + x)$, where $A + x = \{y + x : y \in A\}$. 8

(b) Let $\{E_i\}$ be a sequence of measurable sets. Then show that :

(i) If $E_1 \subset E_2 \subset E_3$ ...., we have $m(\lim E_i) = \lim m(E_i)$

(ii) If $E_1 \supset E_2 \supset E_3$ ...., and $m(E_i) < \infty$ for each i, then we have $m(\lim E_i) = \lim m(E_i)$. 8

2. (c) Show that for any sequence of sets $\{E_i\}$,

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) \qquad 8$$

(d) Let {fn} be a sequence of measurable functions defined on the same measurable set. Then show that :

(i) Sup fn is measurable

(ii) inf fn is measurable

(iii) lim sup fn is measurable

(iv) lim inf fn is measurable. 8

## UNIT—II

3\. (a) Define integral of (i) a simple measurable function (ii) a nonnegative measurable function f. Show that both the definitions coincide for any simple measurable function. 8

(b) Show that if f is an integrable function, then $\left|\int f\,dx\right| \le \int |f|\,dx$. State and prove the condition under which equality in the above inequality occurs. 8

4\. (c) Let f and g be non-negative measurable function. Then show that :

(i) $f \le g \Rightarrow \int f\,dx \le \int g\,dx$

(ii) If $f \le g$ on a measurable set A, then

$$\int_A f\,dx \le \int_A g\,dx$$

(iii) For $a \ge 0$, $\int a f\,dx = a\int f\,dx$

(iv) For measurable sets A and B, if $A \supset B$ then

$$\int_A f\,dx \ge \int_B f\,dx.$$ 8



## UNIT—V

9\. (a) Show that if f, g $\in L^p(\mu)$ and a, b are constants then $af + bg \in L^p(\mu)$. Further show that if $\mu(x) < \infty$, and $0 < p < q < \infty$, then $L^q(\mu) \subset L^p(\mu)$. 8

(b) Let $p \ge 1$ and let f, g $\in L^p(\mu)$. Show that :

$$\left(\int |f+g|^p\,d\mu\right)^{1/p} \le \left(\int |f|^p\,d\mu\right)^{1/p} + \left(\int |g|^p\,d\mu\right)^{1/p}.$$ 8

10\. (c) Let $\psi$ be convex on (a, b) and $a < s < t < u < b$. Show that $\psi(s, t) \le \psi(s, u) \le \psi(t, u)$, where

$$\psi(s, t) = \frac{\psi(t) - \psi(s)}{t - s}, s \ne t.$$ 8

(d) Define $fn \to f$ in measure. Show that if a sequence of measurable functions converges in measure, then the limit function is unique a.e. 8

(d) Let {fn} be a sequence of integrable function such that $\sum_{n=1}^{\infty} \int |fn| \, dx < \infty$. Then show that $\sum_{n=1}^{\infty} fn(x)$ converges a.e., its sum f(x) is integrable and $\int f \, dx = \sum_{n=1}^{\infty} \int fn \, dx$. 8

## UNIT—III

5. (a) Define four derivates of a function f. Find four derivates of the continuous function f at x = 0

$$f(x) = \begin{cases} ax \sin^2 \frac{1}{x} + bx \cos^2 \frac{1}{x} & , \quad x > 0 \\ 0 & , \quad x = 0 \\ a^1 x \sin^2 \frac{1}{x} + b^1 x \cos^2 \frac{1}{x} & , \quad x < 0 \end{cases}$$

where $a < b$, $a^1 < b^1$. 8

(b) If $f \in L(a, b)$, then show that :

(i) $F(x) = \int_a^x f(t) \, dt$ is continuous on [a, b]

(ii) $F \in B \vee [a, b]$. 8

6. (c) Let $f \in B \vee [a, b]$, then show that $f(b) - f(a) = P - N$, and $T = P + N$, all variations being on [a, b], a, b finite. 8

(d) If f is finite valued monotone increasing function defined on finite interval [a, b], then show that $f'$ is measurable and $\int_a^b f' dx \le f(b) - f(a)$. 8

## UNIT—IV

7. (a) Define :

   (i) Ring of sets

   (ii) $\sigma$-ring of sets

   (iii) $\sigma$-algebra of sets.

   Show that every $\sigma$-algebra is a $\sigma$-ring but not conversely. 8

   (b) Let $\mu^*$ be the outer measure on $H(R)$ defined by $\mu$ on R. Show that $\delta^*$ contains $\delta(R)$, the $\sigma$-ring generated by R. 8

8. (c) If $\mu$ is a measure on a ring R, then show that for $A, B \in R, A \subset B$ we have $\mu(A) \le \mu(B)$. 6

   (d) Let $\mu^*$ be an outer measure on $H(R)$ and let $\delta^*$ denote the class of $\mu^*$-measurable sets. Show that $\delta^*$ is a $\sigma$-ring and $\mu^*$ restricted, to $\delta^*$ is a complete measure. 10