AQ–882

M.Sc. Part—I (Semester—II) (CBCS Scheme)
Examination (New)

MATHEMATICS (201)

(Measure and Integration Theory)

Time—Three Hours] [Maximum Marks—80

Note :— Solve **ONE** question from each Unit.

UNIT—I

1. (a) Show that (i) $\beta \subset m$ (ii) β is the σ-algebra generated by each of the following classes, the open intervals, the open sets. Where β is σ-algebra generated by class of intervals of the form $[a, b)$ and m is class of all measurable sets. 8

(b) Show that the following are equivalent :

(i) f is a measurable function

(ii) $\{x; f(x) \geq a\}$ is measurable for every real a

(iii) $\{x; f(x) < a\}$ is measurable for every real a

(iv) $\{x; f(x) \leq a\}$ is measurable for every real a.

Further show that if one of the above is satisfied then $\{x : f(x) = \alpha\}$ is measurable for every extended real number α. 8

2. (c) Show that the class m of measurable sets is σ-algebra. 8

(d) If $\{f_n\}$ is a sequence of measurable functions defined on the same measurable set E. Then show that :

(i) $\sup f_n$ is measurable

(ii) $\inf f_n$ is measurable

(iii) $\lim \sup f_n$ is measurable

(iv) $\lim \inf f_n$ is measurable. 8

UNIT—II

3. (a) Show that if f is a non-negative measurable function, then f = 0 a.e if and only if $\int f\,dx = 0$. 8

(b) Let f and g be non-negative measurable function. Then show that $\int f\,dx + \int g\,dx = \int (f+g)\,dx$. 8

4. (c) Let $\{f_n\}$ be a sequence of measurable function such that $|f_n| \le g$, g is integrable and $\lim f_n = f$ a.e. Then show that f is integrable and $\lim \int fn\,dx = \int f\,dx$. 8

(d) Show that $\lim \int_0^\infty \frac{dx}{\left(1+\frac{x}{n}\right)^n x^{1/n}} = 1$. 8

10. (c) Let $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p(\mu)$, $g \in L^q(\mu)$. Show that $fg \in L^1(\mu)$ and $\int |fg|\,d\mu \le \left(\int |f|^p\,d\mu\right)^{1/p} \left(\int |g|^q\,d\mu\right)^{1/q}$. 8

(d) Define $f_n \to f$ almost uniformly and $f_n \to f$ in measure. Show that if $f_n \to f$ a.u. then $f_n \to f$ in measure. 8

UNIT—III

5. (a) Define four derivatives of a function f and find four derivatives of

$$f(x) = \begin{cases} ax\sin^2\frac{1}{x} + bx\cos^2\frac{1}{x} & x > 0 \\ 0 & x = 0, \\ a'x\sin^2\frac{1}{x} + b'x\cos^2\frac{1}{x} & x < 0 \end{cases}$$

where $a < b$, $a' < b'$. 8

(b) If $f \in L(a, b)$ and $\int_a^x f\,dt = 0$ for all $x \in (a, b)$, then prove that $f = 0$ a.e in (a, b). 8

6. (c) If $a < c < b$, then show that :

$$T_f[a, b] = T_f[a, c] + T_f[c, b].$$ 8

(d) If $F \in L[a, b]$ then show that :

(i) $F(x) = \int_a^x f(t)dt$ is a continuous function on $[a, b]$.

(ii) $F \in B \vee [a, b]$. 8

UNIT—IV

7. (a) If μ is a measure on R, then show that :

(i) $A, B \in R, A \subset B \Rightarrow \mu(A) \le \mu(B)$

(ii) $A, B, C \in R, A, B \subset C, \mu(A) = \mu(C) < \infty$
$\Rightarrow \mu(A \cap B) = \mu(B)$. 8

(b) Let μ^* be the outermeasure on $\mathcal{H}(R)$ defined by μ on R. Show that $\mathcal{S}^*$ contains $\mathcal{S}(R)$, the σ-ring generated by R. 8

8. (c) Define :

(i) ring of sets

(ii) σ-ring of sets

(iii) σ-algebra of sets.

Show that every σ-algebra is a σ-ring but not conversely. 8

(d) If μ is a measure on a ring R and if μ^* is defined on $\mathcal{H}(R)$ by

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty}\mu(E_n) : E_n \in R, n = 1, 2, \ldots, E \subset \bigcup_{n=1}^{\infty} E_n\right\},$$

then show that :

(i) for $E \in R$, $\mu^*(E) = \mu(E)$

(ii) μ^* is an outermeasure on $\mathcal{H}(R)$. 8

UNIT—V

9. (a) Let ψ be convex on (a, b) and $a < s < t < u < b$. Show that $\psi(s, t) \leq \psi(s, u) \leq \psi(t, u)$ where

$$\psi(s, t) = \frac{\psi(t) - \psi(s)}{t - s}\ t \neq s.$$ 8

(b) Show that $\int_0^{\pi} x^{-1/4} \sin x \, dx \leq \pi^{3/4}$. 8