AV–1943

M.Sc. (Semester—II) (CBCS Scheme) Examination

201 : MATHEMATICS

(Measure and Integration Theory)

Time : Three Hours] [Maximum Marks : 80

Note :— Solve **ONE** question from each unit.

UNIT—I

1. (a) Define Lebesgue outer measure. Prove that for any sequence of sets $\{E_i\}$

$$m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i).$$ 2+6

(b) (i) Show that the set of numbers in [0, 1] which possess decimal expansions not containing the digit 5 has measure zero. 4

(ii) Show that if $F \in M$ and $m^*(F \Delta G) = 0$, then G is measurable. 4

2. (c) (i) If f is continuous on E, then show that it is measurable on E. Is its converse true ? Justify. 4

(ii) Let f be a measurable function and B a Borel set; then prove that $f^{-1}(B)$ is a measurable set. 4

(d) If $m^*(E) < \infty$ then prove that E is measurable if and only if, $\forall \in > 0$ there exist disjoint finite intervals $I_1, I_2, ...I_n$ such that $m^*\left(E \Delta \bigcup_{i=1}^{n} I_i\right) < E$. 8

UNIT—II

3. (a) Define integral of :

(i) A simple measurable function, and

(ii) A non-negative measurable function.

Show that the above two definitions coincide for a simple measurable function. 2+2+4

(b) Let f be bounded and measurable on a finite interval [a, b] and let $\in > 0$. Then show that there exists a step function h such that :

$$\int_a^b | f - h | \, dx < \in$$

Apply the result to show :

$$\lim_{\beta \to \infty} \int_a^b f(x) \cdot \sin \beta x \, dx = 0.$$ 8

4. (c) Show that if f is an integrable function, then

$$\left| \int_a^b f dx \right| \leq \int_a^b | f | \, dx$$

When does equality occur ? 6

(d) Show that if f is a non-negative measurable function then f = 0 a.e. if and only if $\int_a^b f\,dx = 0$. 4

(e) Show that :

$$\int_0^1 \frac{x^{1/3}}{1-x}\cdot\log\frac{1}{x}\,dx = g\sum_{n=1}^{\infty}\frac{1}{(3n-1)^2}.$$ 6

UNIT—III

5. (a) Define Dini's four derivatives of a function f. Give an example to show that :

$$D^{+}(f+g) \neq D^{+}_{f} + D^{+}_{g}.$$ 5

(b) Let f be a finite-valued monotone increasing function on [a, b]; then prove that the function f is continuous on a set of points which is almost countable. 6

(c) Show that if f′ exists and is bounded on [a, b] then f ∈ B ∨ [a, b]. 5

6. (d) Show that for a, b finite, the function f ∈ B ∨ [a, b] if and only if f is the difference of two finite valued monotone increasing functions on [a, b]. 8

(e) If f ∈ L(a, b) class of functions integrable on (a, b). Then show that :

(i) $F(x) = \int_a^x f(t)\,dt$ is a continuous function on [a, b].

(ii) F ∈ B ∨ [a, b], class of functions of bounded variation on [a, b]. 8

UNIT—IV

7. (a) Let A, B be subsets of a set C, let A, B, C ∈ ℝ and let μ be the measure on ℝ. Show that if μ(A) = μ(C) < ∞ then μ(A ∩ B) = μ(B). 6

(b) Prove that the completion of a σ-finite measure is σ-finite. 4

(c) For a ring ℝ prove that :

$$\mathcal{H}(\mathbb{R}) = \left[E : E \subset \bigcup_{n=1}^{\infty} E_n,\ E_n \in \mathbb{R}\right].$$ 6

8. (d) Let μ* be an outer measure on $\mathcal{H}(\mathbb{R})$, and let $\mathbf{S}^*$ denote the class of μ*-measurable sets, then prove that $\mathbf{S}^*$ is a σ-ring and μ* restricted to $\mathbf{S}^*$ is a complete measure. 8

(e) If μ is a σ-finite measure on a ring ℝ, then prove that it has a unique extension to the σ-ring $\mathcal{S}(\mathbb{R})$. 8

UNIT—V

9. (a) Define a convex function. Let ψ be defined on (a, b). Then prove that ψ is convex on (a, b) if and only if, for each x and y such that $a < x < y < b$, the graph of ψ on (a, x) and (y, b) does not lie below the line through X and Y axes. 6

(b) Let $\{f_n\}$ be a sequence in $L^\infty(\mu)$ such that $\|f_n - f_m\|_\infty \to 0$ as $n, m \to \infty$. Then prove that there exists a function f such that $\lim f_n = f$ a.e. Also show that $f \in L^\infty(\mu)$ and $\lim \|f_n - f\|_\infty \to 0$. 10

10. (c) (i) If $f_n \to f$ in the mean of order p $(p > 0)$ then prove that $f_n \to f$ in measure. 4

(ii) $f_n \to f$ a.u. (almost uniformly) then prove that $f_n \to f$ a.e. 4

(d) State and prove Holder's Inequality. 8

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