AV–1947

M.Sc. Part—I Semester-II (C.B.C.S. Scheme) Examination

MATHEMATICS

(Riemannian Geometry)

Paper—205

Time : Three Hours] [Maximum Marks : 80

Note :— Solve **FIVE** questions, selecting **ONE** from each unit.

UNIT—I

1. (a) Compute the quantities g^{mn} for a V_3-space whose fundamental form in coordinates u, v, w is :

$$adu^2 + bdv^2 + cdw^2 + 2fdvdwt + 2kdwdn + 2hdudv.$$ 8

(b) Show that under the change of the coordinate system the Christoffel symbols transform as :

$$[mn, r]' = [ab, c] \frac{\partial x^a}{\partial x'^m} \frac{\partial x^b}{\partial x'^n} \frac{\partial x^c}{\partial x'^r} + g_{ab} \frac{\partial x^a}{\partial x'^r} \frac{\partial^2 x^b}{\partial x'^m \partial x'^n}.$$ 8

2. (c) Define absolute derivative. Also show that $\frac{\delta T^r}{\delta u}$ is a contravarient vector. 1+7

(d) Show that in :

$$A^{mn}_{;n} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^n} (A^{mn} \sqrt{g}) + A^{nr} \Gamma^m_{nr}$$

the last term vanishing if A^{nr} is Skew-Symmetric. Also show that :

$$A^n_{m;n} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^n} (A^n_m \sqrt{g}) - A^n_r \Gamma^r_{mn}.$$ 8

UNIT—II

3. (a) Show that a vector $\overline{T}$ of variable magnitude undergoes a parallel displacement along a curve C if the absolute derivative $\overline{T}$ along C has the same direction of $\overline{T}$ at each point of C :

$$\text{i.e. } T^i_{;k} \frac{dx^k}{d^s} = T^i g(s).$$ 8

(b) Define Geodesics and obtain its differential equation. 8

4. (c) Necessary and sufficient conditions that a system of coordinates be geodesic with pole at O are that their second covariant derivatives with respect to the metric of the space V_N all vanish at that point. Prove this. 8

(d) Show that at the origin of a Riemannian coordinate system y^i :

$$\frac{\partial}{\partial y^m}\Gamma^i_{ki} = \frac{\partial}{\partial y^k}\Gamma^i_{mi}, \; \neq k, m. \quad 8$$

UNIT—III

5. (a) Obtain a formula for covariant tensor from R^p_{rmn} as :

$$R_{prmn} = g_{ps}R^s_{.rmn} = \frac{1}{2}(g_{pn,rm} + g_{rm,pn} - g_{pm,rn} - g_{rn,pm}) + g^{sq}([pn,s][rm,q] - [pm,s][rn,q])$$

8

(b) For the metric in spherical polar coordinates :

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

calculate the non-zero components R_{prmn}. 8

6. (c) Show that :

(i) $R^p_{.rmn} + R^p_{.mnr} + R^p_{.nrm} = 0$

(ii) $R_{prmn} + R_{pmnr} + R_{pnrm} = 0$ 8

(d) List the independent components of Riemann tensor R_{rpmn} for the Riemannian space V_N, N = 2, 3, 4. 8

UNIT—IV

7. (a) Define Ricci tensor and Einstein tensor and show that they are symmetric tensor. 2+2+2+2

(b) Let $T^r(u, v)$ be a vector field over a two space V_2 with equation $x^r = x^r(u, v)$ immersed in a Riemannian space V_N. Then prove that :

$$\frac{\delta^2 T^r}{\delta u \delta v} - \frac{\delta^2 T^r}{\delta v \delta u} = R^r_{.ymn} T^p \frac{\partial x^m}{\partial u}\frac{\partial x^n}{\partial v}. \quad 8$$

8. (c) On a surface of a sphere of radius a, find the curvature tensor, the Ricci tensor and the curvature invariant (in polar coordinates). 8

(d) State and prove Bianchi Identity. 8

UNIT—V

9. (a) The Riemann curvature is independent of the pair of orthogonal unit vectors in V_2. Prove this. 8

(b) Show that every Riemannian space V_2 is an Einstein space. 8

10. (c) Prove that, if $R^a_i = g^{am} R_{im}$, then $R^a_{i;a} = \frac{1}{2}\frac{\partial R}{\partial x^i}$. 8

(d) Prove that if a Riemannian space V_n is isotropic at each point of a region, then the Riemannian curvature is constant throughout the region. 8