AV–1944

M.Sc. (Part—I) Semester—II (CBCS Scheme) Examination

202 : MATHEMATICS

(Advanced Linear Algebra and Field Theory)

Time : Three Hours] [Maximum Marks : 80

Note :— Solve any **ONE** question from each unit.

UNIT—I

1. (a) Determine the characteristic roots and the corresponding characteristic vectors of the matrix :

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ 3+5=8

(b) Define : Minimal polynomial. Prove that the minimal polynomial of a matrix is a divisor of every polynomial that annihilates this matrix. 1+7=8

2. (c) Show that the matrix

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

is diagonalizable. Also find the diagonal form and diagonalizing matrix P. 4+4=8

(d) Find all possible Jordan Canonical Forms of matrices whose characteristic polynomial p(x) and minimal polynomial m(x) are as follows :

(i) $p(x) = (x-2)^3(x-5)^4$, $m(x) = (x-2)^2(x-5)^2$

(ii) $p(x) = (x-3)^6$, $m(x) = (x-3)^2$. 4+4=8

UNIT—II

3. (a) Prove that the relation of 'Congruence of matrices' is an equivalence relation in the set of all $n \times n$ matrices over a field F. 8

(b) Prove that a real symmetric matrix is positive definite iff all its eigen values are positive. 8

4. (c) Show that the quadratic form $6x^2 + 17y^2 + 3z^2 - 20xy - 14yz + 8zx$ in three variables is positive semi-definite. 8

(d) Prove that the number of positive terms in any two normal reductions of a real quadratic form is the same. 8

UNIT—III

5. (a) Show that :

(i) $x^3 + 3x + 2 \in \mathbf{Z}/(7)[x]$ is irreducible over the field $\mathbf{Z}/(7)$.

(ii) $x^4 + 8 \in Q[x]$ is irreducible over Q. 4+4=8

(b) Let $F \subseteq E \subseteq K$ be fields. If $[K : E] < \infty$ and $[E : F] < \infty$ then prove that :

(i) $[K : F] < \infty$

(ii) $[K : F] = [K : E] [E : F]$. 4+4=8

6. (c) Determine all (i) quadratic (ii) cubic irreducible polynomials over $\mathbf{Z}/(2)$. 4+4=8

(d) Let F be a field. Prove that there exists an extension $\overline{F}$ that is algebraic over F and is algebraically closed; that is, each field has an algebraic closure. 8

UNIT—IV

7. (a) Prove that the splitting field of $f(x) = x^4 - 2 \in Q[x]$ over Q is $Q(2^{1/4}, i)$ and its degree of extension is 8. 5+3=8

(b) Prove that if $f(x) \in F[x]$ is irreducible over F, then all roots of f(x) have the same multiplicity. 8

8. (c) Prove that the multiplicative group of non-zero elements of a finite field is cyclic. 8

(d) Let K be a splitting field of the polynomial $f(x) \in F[x]$ over a field F. If E is another splitting field of f(x) over F, then prove that there exists an isomorphism $\sigma : E \to K$ that is identity on F. 8

UNIT—V

9. (a) Prove that following are equivalent statements :

(i) $a \in \mathbb{R}$ is constructible from Q.

(ii) (a, 0) is a constructible point from $Q \times Q$.

(iii) (a, a) is a constructible point from $Q \times Q$.

(iv) (0, a) is a constructible point from $Q \times Q$. 8

(b) Prove that every polynomial $f(x) \in \mathbb{C}[x]$ factors into linear factors in $\mathbb{C}[x]$. 8

10. (c) Let F be a field, and let U be a finite subgroup of the multiplicative group $F^* = F - \{0\}$. Then prove U is cyclic. 8

(d) Prove that the group $G(Q(\alpha)/Q)$, where $\alpha^5 = 1$ and $\alpha \neq 1$, is isomorphic to the cyclic group of order 4. 8