UNIT—V

9. (a) If S and T are two bounded self-adjoint positive operators on a Hilbert space H such that ST = TS and if

$S_1 = \dfrac{S}{\|S\|}, S_{n+1} = S_n - S_n^2, n \geq 1$, then show that $0 \leq S_n \leq I$. 8

(b) Show that a bounded operator $P : H \to H$ on a Hilbert space H is a projection if and only if P is self-adjoint and $P^2 = p$. 8

10. (c) Suppose T is a positive bounded self-adjoint operator on a Hilbert space H.

Put $A_0 = 0$ and $A_{n+1} = A_n + \frac{1}{2}(T - A_n^2), n \geq 0$.

Show that

(i) $A_n \leq I, n \geq 0,$

(ii) $A_n \leq A_{n+1}, n \geq 0.$ 8

(d) Prove that on a Hilbert space H, the following hold :

(i) $P = P_1P_2$ is a projection on H if and only if P_1, P_2 commute. In that case P projects H onto $P_1(H) \cap P_2(H)$.

(ii) Two closed subspaces Y and V of H are orthogonal if and only if $P_yP_v = 0$. 8

AQ–1058

M.Sc. Part-II (Semester—IV) (CBCS) Examination

MATHEMATICS

(Functional Analysis—II)

Paper—401

Time : Three Hours] [Maximum Marks : 80

Note :— Solve **ONE** question from each Unit.

UNIT—I

1. (a) Show that for a bounded linear operator $T : H_1 \to H_2$, H_1, H_2 being Hilbert spaces, Hilbert-adjoint operator $T^* : H_2 \to H_1$ exists, is unique and is a bounded linear operator with $\| T^* \| = \| T \|$. 8

(b) Show that every Hilbert space H is reflexive. 8

2. (c) If $h : H_1 \times H_2 \to K$ is a bounded sesquilinear form, H_1, H_2 being Hilbert spaces, then show that there is a bounded linear operator $S : H_1 \to H_2$ such that :

(i) $h(x, y) = \langle Sx, y\rangle, x \in H_1, y \in H_2$

(ii) S is uniquely determined by h

(iii) $\|h\| = \|S\|$. 8

(d) For a bounded linear operator $T : H \to H$, H being Hilbert space, show that :

(i) If T is self-adjoint, $\langle Tx, x\rangle$ is real for all $x \in H$.

(ii) If H is complex and $\langle Tx, x\rangle$ is real for all $x \in H$, then T is self-adjoint. 8

UNIT—II

3. (a) Show that the resolvent set $\rho(T)$ of a bounded linear operator on a complex Banach space is open. 8

(b) For $T \in B(x, x)$, x being a complex Banach space and $p(\lambda) = a_n \lambda^n + \ldots\ldots + a_0$, $a_n \neq 0$, show that $p(\sigma(T)) = \sigma(p(T))$. 8

4. (c) For $T \in B(x, x)$, x being a complex Banach space and $\lambda, \mu \in \rho(T)$. Show that :

(i) The resolvent R_λ of T satisfies $R_\mu - R_\lambda = (\mu - \lambda) R_\mu R_\lambda$.

(ii) R_λ commutes with every $S \in B(x, x)$ which commutes with T.

(iii) $R_\lambda R_\mu = R_\mu R_\lambda$. 8

(d) If $X \neq \{0\}$ is a complex Banach space and $T \in B(x, x)$, show that $\rho(T) \neq \phi$. 8

UNIT—III

5. (a) If $\{T_n\}$ is a sequence of compact linear operators from a normed space X into a Banach space Y, show that if $\{T_n\}$ is uniformly operator convergent, then the limit operator is compact. 8

(b) Show that the set of eigenvalues of a compact linear operator $T : X \to X$, X being normed space, is countable (may be finite or empty) and the only possible point of accumulation is $\lambda = 0$. 8

6. (c) Prove compactness of $T : l^2 \to l^2$ defined by

$$y = (\eta_j) = Tx, \text{ where } \eta_j = \frac{\xi_j}{j^2}, j \geq 1, x = (\xi_j). \quad 8$$

(d) Show that the range $R(T)$ of a compact linear operator $T : X \to Y$ is separable. Here X and Y are normed spaces. 8

UNIT—IV

7. (a) Show that the spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T : H \to H$ on a complex Hilbert space H is real. 8

(b) Define residual spectrum of a bounded linear operator on a complex Hilbert space H. Show that the residual spectrum of a bounded self-adjoint operator on a complex Hilbert space H is empty. 8

8. (c) If T is a bounded self-adjoint operator on a complex Hilbert space H and $m = \inf_{\|x\|=1} \langle Tx, x\rangle$, $M = \sup_{\|x\|=1} \langle Tx, x\rangle$ show that $\sigma(T) \subset [m, M]$. 8

(d) If T is a bounded self-adjoint operator on a complex Hilbert space H and

$m = \inf_{\|x\|=1} \langle Tx, x\rangle$, $M = \sup_{\|x\|=1} \langle Tx, x\rangle$ show that

$\|T\| = \max\{|m|, |m|\} = \sup_{\|x\|=1} |\langle Tx, x\rangle|$. 8